Sampling equilibrium through descriptive simulations

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Presentation overview

• Sampling equilibrium and sampling dynamics
  – example: coordination game
• Descriptive simulation
  – environment and players
  – example: convergence & equilibrium
• Mathematics behind ...
  – mathematical model of a player
  – average dynamics of simulation
  – equilibrium
  – example revisited
• Conclusions
**Procedurally rational players**

**Sampling equilibrium [Osbourne, Rubinstein, 1998]**
- Distribution of strategies $x = (x_1, \ldots, x_S) \in \Delta_S$
- Winning probability $w(x) = (w(1, x), \ldots, w(S, x))$
- Sampling equilibrium $x^* \in \Delta_S$

\[ \forall j \in S \quad x_j^* = w(j, x^*) \]

- For any normal form game, there exists a sampling equilibrium.

**Sampling dynamics [Sethi, 2000]**
- Evolutionary style dynamics $\dot{x}_j = x_j - w(j, x)$, for all $j \in \Delta_S$
- Sampling dynamics is well defined:
  - A simplex $\Delta_S$ is forward invariant.
  - A distribution $x^*$ is a sampling equilibrium if and only if it is a stationary point of sampling dynamics.
Example

Coordination game

Winning probability

\[ w(1, x) = x_1 \]

Sampling dynamics

\[ \dot{x} = 0 \]

Sampling equilibrium

Any distribution \( x \in \Delta_S \) is a sampling equilibrium.

There is a fraction of players using the second pure strategy, due to the sampling procedure. Hence, the winning probability for the second pure strategy is always positive.
**Simulation: overview**

**Simulation**

Environment

Game

1   i   ...   j   n

The simulation comprises an environment object and \( n \) player objects.

There are random delays and endogenous noise.

**Assumptions**

- The sole purpose of the environment is to provide usual “random matching” evolutionary scenario.
- The players are anonymous to the environment and to each other.

**Consequences**

- The sampling players may be matched.
- A player may start sampling at time \( t \), continue sampling at some later time \( t + l_1 \), and finish it at some time \( t + l_1 + \ldots + l_S \), where \( l_i > 0, i = 1, \ldots, S \).
Simulation: environment

Main loop of the simulation

Start

Choose at random $k/2$ pairs of players

Receive strategies from chosen players

Calculate payoffs and send them to players

$t=0$

$t++$

$t<T?$

Stop
Simulation: player

1. S=2, AS=2
   - send AS

2. receive P2
   - S=0

3. if P1 > P2?
   - yes: AS=1 send AS
   - no: Stop

4. S->0?
   - yes: S=1, AS=1
   - no: S->1?

   if S->1?
     - yes: send AS
     - no: Stop

   if S->0?
     - yes: S=2
     - no: Stop
Results of the simulation

Results for a fraction of passive players using the first pure strategy.

\[ n = 100, \ k = 2, \ \delta = 1/10 \]

\[ n = 100, \ k = 100, \ \delta = 1/10 \]

The average fraction of passive players using the first pure strategy converges.
Players, mathematics and Markov chains

Mathematical model of a player

The transition matrix $P$

$$
egin{bmatrix}
1 - \delta & \delta & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 1 \\
1 - \delta & \delta & 0 & \cdots & 0
\end{bmatrix}
$$

The modified transition matrix $\tilde{P}$

$$
\tilde{P} = \frac{k}{n}P + \left(1 - \frac{k}{n}\right)I_{(S+1)\times(S+1)}
$$

Properties

- Markov chain $P$ is ergodic
- Invariant measure $\pi$ reads
  $$
  \pi = \left(\pi_0, \pi_1, \ldots, \pi_S\right)
  $$
  $$
  \pi_0 = \frac{1 - \delta}{1 + \delta(S - 1)}
  $$
  $$
  \pi_j = \frac{\delta}{1 + \delta(S - 1)}
  $$
- It works for the modified chain $\tilde{P}$
  $$
  \pi^T \cdot \tilde{P} = \pi^T
  $$
- Distribution of players among nodes is $n\pi$
Description of average paths

Inflow – outflow

Outflow term

$$-\frac{k}{n} \delta \pi_0 n x_j(t)$$

Inflow term

$$+\frac{k}{n} (1 - \delta) \pi_S n w(j, \Phi_t)$$

$$\Phi_t = (\phi_0, \ldots, \phi_t), \quad \phi_t = \pi_0 x_t + \pi_a$$

Difference equation describing behavior of an average path

$$\pi_0 n x_j(t + 1) = \pi_0 n x_j(t) - \frac{k}{n} \delta \pi_0 n x_j(t) + \frac{k}{n} (1 - \delta) \pi_S n w(j, \Phi_t)$$

$$x_j(t + 1) = x_j(t) - \frac{k}{n} \delta x_j(t) + \frac{k}{n} \delta w(j, \Phi_t)$$

(1)
Sampling equilibrium revisited

Sampling dynamics
- A simplex $\triangle_S$ is forward invariant under dynamics (1).

Sampling equilibrium
- A distribution $x^\diamond \in \triangle_S$ is a sampling equilibrium if and only if it is a stationary point of the dynamics (1), i.e. $x^\diamond(t + 1) = x^\diamond(t)$ for all $t$.
- A distribution $x^\diamond$ satisfies
  \[ x^\diamond_j = w(j, \pi_0 x^\diamond + \pi_a), \quad \text{for all } j \in S. \]
- For any normal form game, there exists a sampling equilibrium $x^\diamond \in \triangle_S$. 
Sampling dynamics and lags: two strategies case

Possible lags

Incoming paths

(0, ..., 2, 0)

Incoming \((l_1, l_2)\)-paths

\[l_1\] times

\((0, \ldots, 0 \text{ or } 2, \underbrace{1, \ldots, 1}_{l_1 \text{ times}}, 2, \ldots, 2, 0)\).

\[l_2\] times

Distribution of lags

- Let \(A\) denote a set of all incoming paths at time \(t + 1\).
- Let \(B(l_1, l_2)\) denote a set of all incoming \((l_1, l_2)\)-paths at time \(t + 1\).
- Distribution of lags is

\[
\theta(l_1, l_2) = P(B(l_1, l_2) | A)
\]
Distribution of lags

Conditional probability (two strategies)

\[ \theta(l_1, l_2) = \frac{\left[ e^{1\bar{p}t-l_1-l_2} \right]_2 \tilde{p}(2, 1) + \left[ e^{1\bar{p}t-l_1-l_2} \right]_0 \tilde{p}(0, 1) \tilde{p}(1, 1)^{l_1-1} \tilde{p}(1, 2) \tilde{p}(2, 2)^{l_2-1} \tilde{p}(2, 0)}{\left[ e^{1\bar{p}t} \right]_2 \tilde{p}(2, 0)} \]

Closed-form formula (two strategies)

\[ \theta(l_1, l_2) = -\frac{k^2 (1 - \frac{k}{n})^{l_1+l_2} \delta \left( \frac{n-k(\delta+1)}{n} \right)^{-l_1-l_2} \left( \frac{n-k(\delta+1)}{n} \right)^{l_1+l_2} + \delta \left( \frac{n-k(\delta+1)}{n} \right)^t}{(k-n)^2 \left( (1 - \frac{k}{n})^t - \left( \frac{n-k(\delta+1)}{n} \right)^t + \left( (1 - \frac{k}{n})^t - 1 \right) \delta \right)} \]

General asymptotic distribution

\[ \theta(l_1, \ldots, l_S) = \left( \frac{k}{n-k} \right)^S \prod_{j \in S} \left( 1 - \frac{k}{n} \right)^{l_j} \]
Starting example once again

Dynamics

\[ x(t + 1) = \left(1 - \frac{k\delta}{n}\right) x(t) + \frac{k\delta}{n} \left(\pi_1 + \pi_0 \sum_{l_1 + l_2 > 1} \theta(l_1, l_2)x(t + 1 - l_1 - l_2)\right) \]

Solution

• There is a unique sampling equilibrium \( x^{\diamond} = 1/2 \).
• The equilibrium is globally asymptotically stable.

These results fit the results of the simulation.
Conclusions

General

• The very process of building a descriptive simulation reveals certain technical details concerning a model. These should be taken into account while developing a mathematical model.
• The simulation is a benchmark for a mathematical model (validation of a mathematical model).

Specific – sampling equilibrium

• The technical details led to a model with endogenous noise and lags.
• The inclusion of noise removed problems previously encountered with the coordination game.
Technicalities

Simulation

- The simulation was implemented in the R language.
  http://www.r-project.org
- The code of the simulation is available at
  http://akson.sgh.waw.pl/~mramsz
Thank You