Sampling equilibrium through descriptive simulations

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Presentation overview

- Sampling equilibrium and sampling dynamics
  - example: coordination game
- Descriptive simulation
  - environment and players
  - example: convergence & equilibrium
- Mathematics behind ...
  - mathematical model of a player
  - average dynamics of simulation
  - equilibrium
  - example revisited
- Conclusions

Procedurally rational players

Sampling equilibrium [Osbourne, Rubinstein, 1998]

- Distribution of strategies $x = (x_1, \ldots, x_S) \in \triangle_S$
- Winning probability $w(x) = (w(1, x), \ldots, w(S, x))$
- Sampling equilibrium $x^\star \in \triangle_S$

$$\forall_{j \in \mathcal{S}} \quad x_j^\star = w(j, x^\star)$$

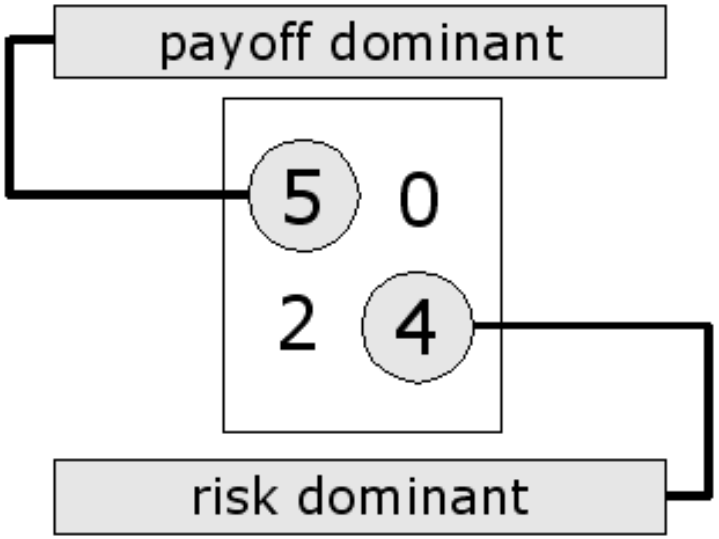
- For any normal form game, there exists a sampling equilibrium.

Sampling dynamics [Sethi, 2000]

- Evolutionary style dynamics $\dot{x}_j = x_j - w(j, x)$, for all $j \in \triangle_S$
- Sampling dynamics is well defined:
  - A simplex $\triangle_S$ is forward invariant.
  - A distribution $x^\star$ is a sampling equilibrium if and only if it is a stationary point of sampling dynamics.

Example

Coordination game



Winning probability

$$w(1, x) = x_1$$

Sampling dynamics

$$\dot{x} = 0$$

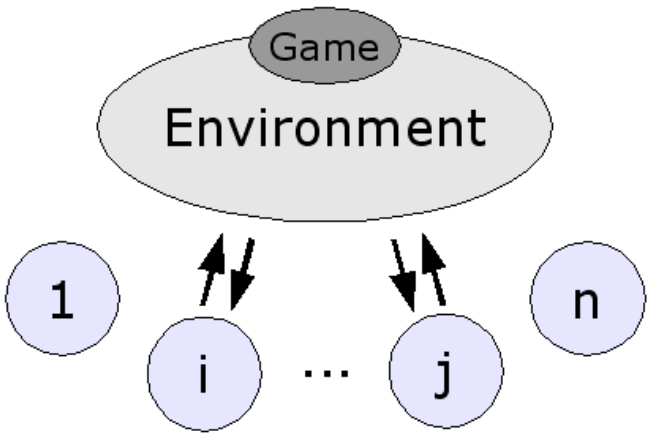
Sampling equilibrium

Any distribution $x \in \triangle_S$
is a sampling equilibrium.

There is a fraction of players using the second pure strategy, due to the sampling procedure. Hence, the winning probability for the second pure strategy is **always** positive.

Simulation: overview

Simulation



The simulation comprises an environment object and n player objects.

There are random delays and endogenous noise.

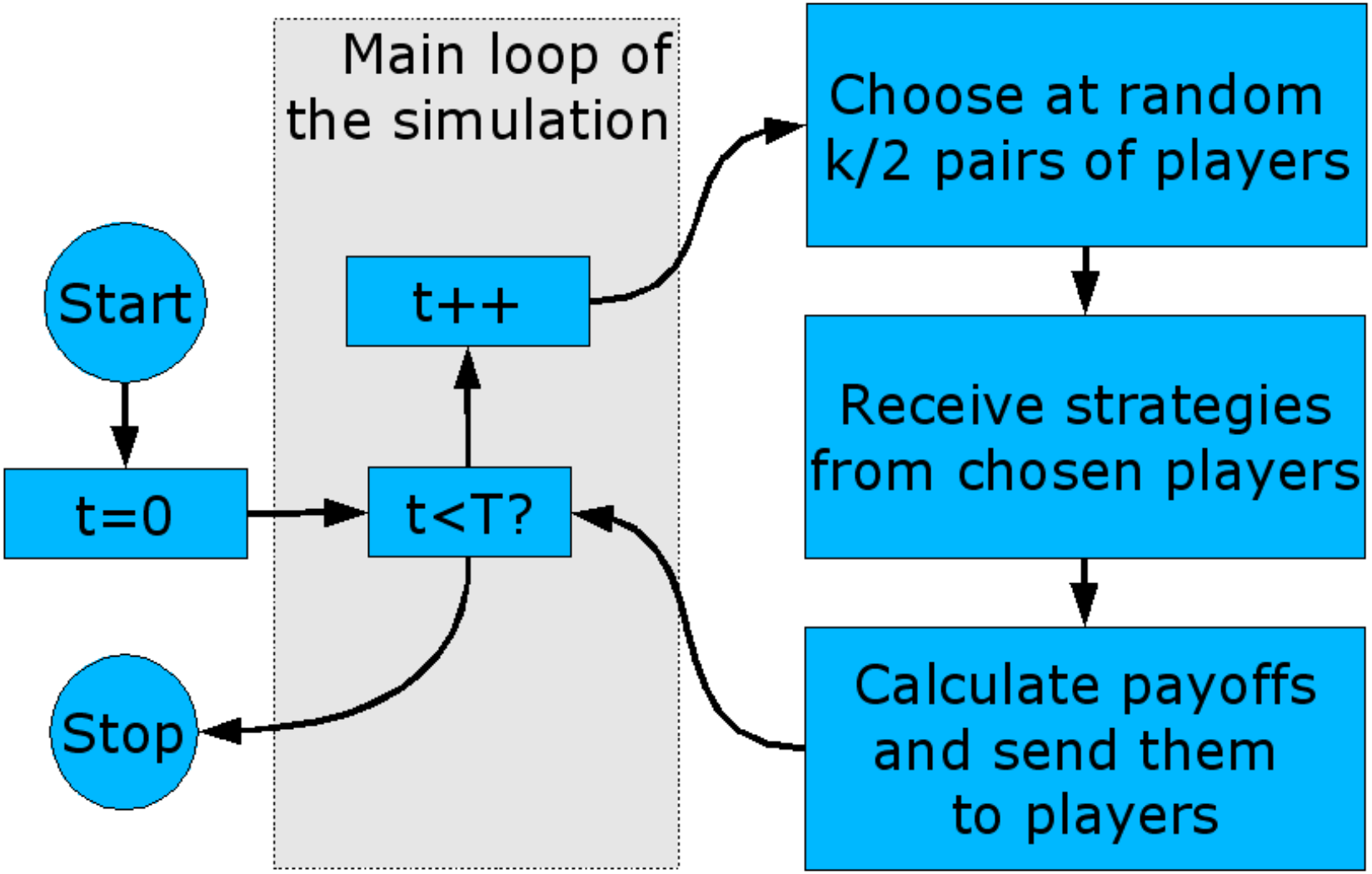
Assumptions

- The sole purpose of the environment is to provide usual "random matching" evolutionary scenario.
- The players are anonymous to the environment and to each other.

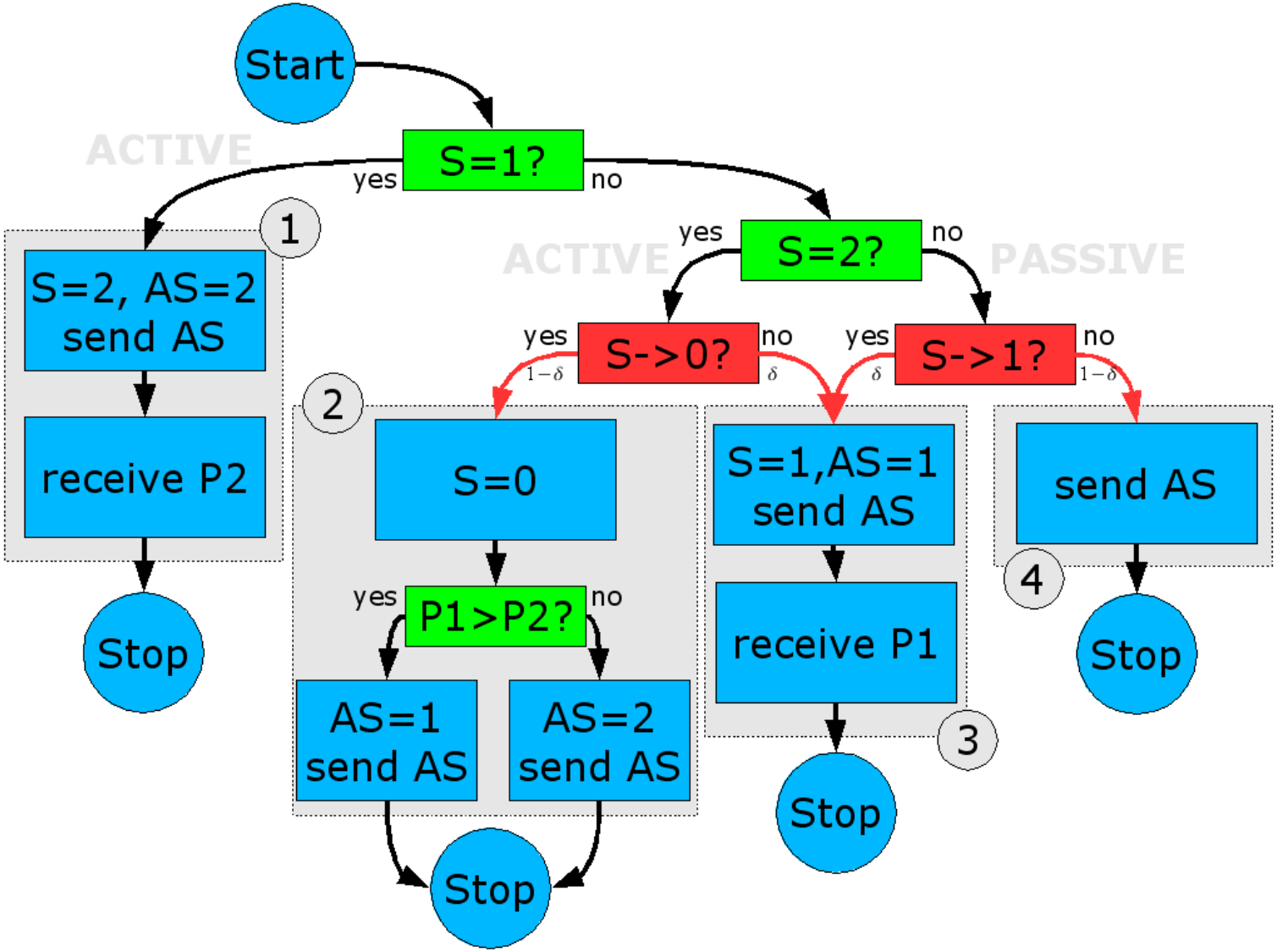
Consequences

- The sampling players may be matched.
- A player may start sampling at time t, continue sampling at some later time $t + l_1$, and finish it at some time $t + l_1 + \ldots + l_S$, where $l_i > 0$, $i = 1, \ldots, S$.

Simulation: environment

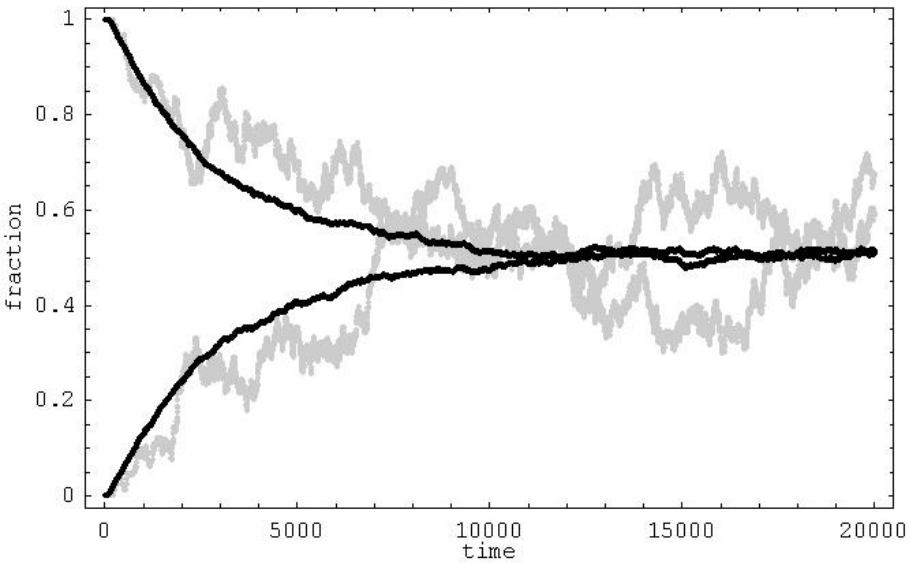


Simulation: player

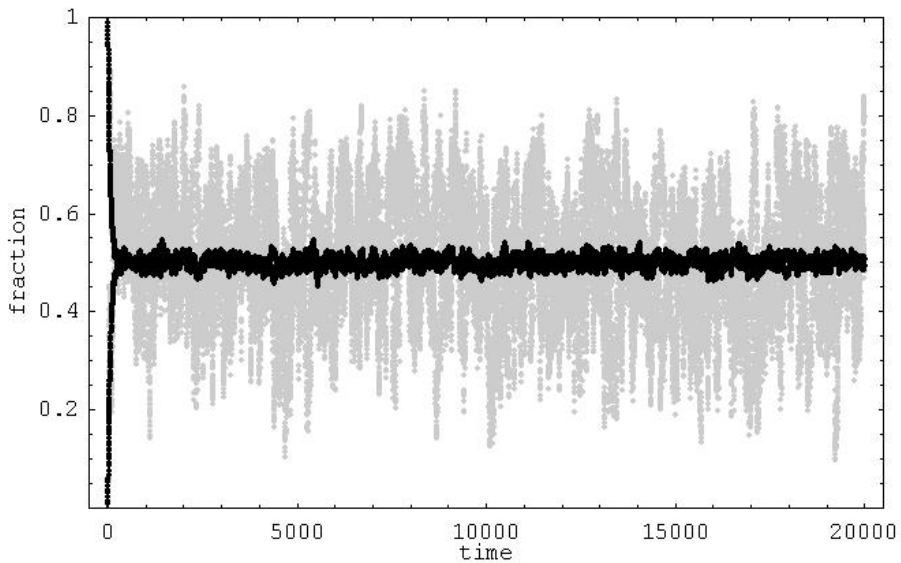


Results of the simulation

Results for a fraction of passive players using the first pure strategy.



$n = 100,\ k = 2,\ \delta = 1/10$

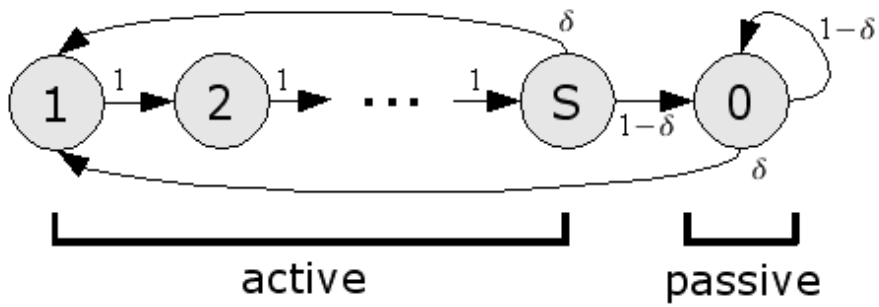


$n = 100,\ k = 100,\ \delta = 1/10$

The average fraction of passive players using the first pure strategy converges.

Players, mathematics and Markov chains

Mathematical model of a player



The transition matrix $\mathcal{P}$

$$\begin{bmatrix} 1-\delta & \delta & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & \cdots & 1 \\ 1-\delta & \delta & 0 & \cdots & 0 \end{bmatrix}$$

The modified transition matrix $\tilde{\mathcal{P}}$

$$\tilde{\mathcal{P}} = \frac{k}{n}\mathcal{P} + \left(1 - \frac{k}{n}\right) \mathrm{I}_{(S+1)\times(S+1)}$$

Properties

- Markov chain $\mathcal{P}$ is ergodic
- Invariant measure π reads

$$\pi = (\pi_0, \underbrace{\pi_1, \ldots, \pi_S}_{\pi_a})$$

$$\pi_0 = \frac{1-\delta}{1+\delta(S-1)}$$

$$\pi_j = \frac{\delta}{1+\delta(S-1)}$$

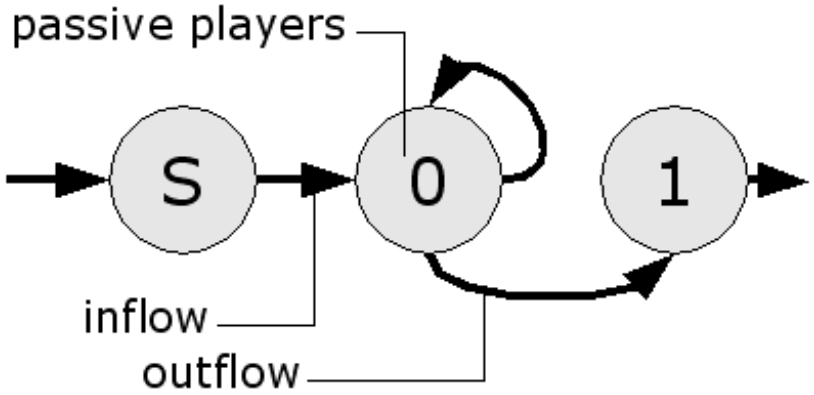
- It works for the modified chain $\tilde{\mathcal{P}}$

$$\pi^T \cdot \tilde{\mathcal{P}} = \pi^T$$

- Distribution of players among nodes is $n\pi$

Description of average paths

Inflow − outflow



Outflow term

$$-\frac{k}{n}\,\delta\,\pi_0\,n\,x_j(t)$$

Inflow term

$$+\frac{k}{n}\,(1-\delta)\,\pi_S\,n\,w(j,\Phi_t)$$

$$\Phi_t = (\phi_0,\ldots,\phi_t), \qquad \phi_t = \pi_0\,x_t + \pi_a$$

Difference equation describing behavior of an average path

$$\pi_0\,n\,x_j(t+1) = \pi_0\,n\,x_j(t) - \frac{k}{n}\,\delta\,\pi_0\,n\,x_j(t) + \frac{k}{n}\,(1-\delta)\,\pi_S\,n\,w(j,\Phi_t)$$

$$x_j(t+1) = x_j(t) - \frac{k}{n}\,\delta\,x_j(t) + \frac{k}{n}\,\delta\,w(j,\Phi_t) \tag{1}$$

Sampling equilibrium revisited

Sampling dynamics

- A simplex $\triangle_S$ is forward invariant under dynamics (1).

Sampling equilibrium

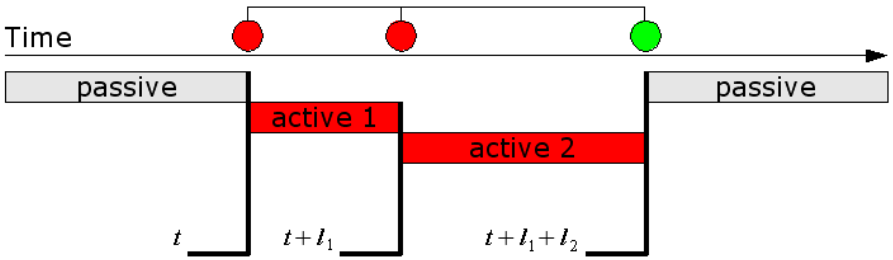
- A distribution $x^\diamondsuit \in \triangle_S$ is a sampling equilibrium if and only if it is a stationary point of the dynamics (1), i.e. $x^\diamondsuit(t+1) = x^\diamondsuit(t)$ for all t.
- A distribution $x^\diamondsuit$ satisfies

$$x_j^\diamondsuit = w(j, \pi_0\, x^\diamondsuit + \pi_a), \quad \text{for all } j \in \mathcal{S}.$$

- For any normal form game, there exists a sampling equilibrium $x^\diamondsuit \in \triangle_S$.

Sampling dynamics and lags: two strategies case

Possible lags



Incoming paths

$$(0, \ldots, 2, 0)$$

Incoming (l_1, l_2)-paths

$$(0, \ldots, 0 \text{ or } 2, \overbrace{1, \ldots, 1}^{l_1 \text{ times}}, \underbrace{2, \ldots, 2}_{l_2 \text{ times}}, 0).$$

Distribution of lags

- Let A denote a set of all incoming paths at time $t+1$.
- Let $B(l_1, l_2)$ denote a set of all incoming (l_1, l_2)-paths at time $t+1$.
- Distribution of lags is

$$\theta(l_1, l_2) = \mathsf{P}(B(l_1, l_2)|A)$$

Distribution of lags

Conditional probability (two strategies)

$$\theta(l_1, l_2) = \frac{\left(\left[\mathrm{e}^1 \tilde{\mathcal{P}}^{t-l_1-l_2}\right]_2 \tilde{p}(2,1) + \left[\mathrm{e}^1 \tilde{\mathcal{P}}^{t-l_1-l_2}\right]_0 \tilde{p}(0,1)\right) \tilde{p}(1,1)^{l_1-1} \tilde{p}(1,2) \tilde{p}(2,2)^{l_2-1} \tilde{p}(2,0)}{\left[\mathrm{e}^1 \tilde{\mathcal{P}}^{t}\right]_2 \tilde{p}(2,0)}$$

Closed-form formula (two strategies)

$$\theta(l_1, l_2) = -\frac{k^2 \left(1 - \frac{k}{n}\right)^{l_1+l_2} \delta \left(\frac{n-k(\delta+1)}{n}\right)^{-l_1-l_2} \left(\left(\frac{n-k(\delta+1)}{n}\right)^{l_1+l_2} + \delta \left(\frac{n-k(\delta+1)}{n}\right)^{t}\right)}{(k-n)^2 \left(\left(1 - \frac{k}{n}\right)^{t} - \left(\frac{n-k(\delta+1)}{n}\right)^{t} + \left(\left(1 - \frac{k}{n}\right)^{t} - 1\right) \delta\right)},$$

General asymptotic distribution

$$\theta(l_1, \ldots, l_S) = \left(\frac{k}{n-k}\right)^S \prod_{j \in \mathcal{S}} \left(1 - \frac{k}{n}\right)^{l_j}$$

Starting example once again

Dynamics

$$x(t+1) = \left(1 - \frac{k\delta}{n}\right) x(t) + \frac{k\delta}{n} \left(\pi_1 + \pi_0 \sum_{l_1+l_2>1} \theta(l_1, l_2) x(t+1-l_1-l_2)\right)$$

Solution

- There is a unique sampling equilibrium $x^{\diamondsuit} = 1/2$.
- The equilibrium is **globally asymptotically stable**.

These results fit the results of the simulation.

Conclusions

General

- The very process of building a descriptive simulation reveals certain technical details concerning a model. These should be taken into account while developing a mathematical model.
- The simulation is a benchmark for a mathematical model (validation of a mathematical model).

Specific – sampling equilibrium

- The technical details led to a model with endogenous noise and lags.
- The inclusion of noise removed problems previously encountered with the coordination game.

Technicalities

Simulation

- The simulation was implemented in the **R** language.

  `http://www.r-project.org`
- The code of the simulation is available at

  `http://akson.sgh.waw.pl/~mramsz`

Thank You